PHYSICAL REVIEW A VOLUME 16, NUMBER 2 AUGUST 1977

Large-distance and long-time properties of a randomly stirred fluid

Dieter Forster*
Department of Physics, Temple University, Philadelphia, Pennsylvania 19122

David R. Nelson†
Department of Physics, Harvard University, Cambridge, Massachusetts 02138

Michael J. Stephen‡
Physics Department, Rutgers University, New Brunswick, New Jersey 08903
(Received 14 February 1977)

Dynamic renormalization-group methods are used to study the large-distance, long-time behavior of velocity correlations generated by the Navier-Stokes equations for a randomly stirred, incompressible fluid. Different models are defined, corresponding to a variety of Gaussian random forces. One of the models describes a fluid near thermal equilibrium, and gives rise to the usual long-time tail phenomena. Apart from simplifying the derivation of the latter, our methods clearly establish their universality, their connection with Galilean invariance, and their analytic form in two dimensions, $\sim(\log t)^{-1/2}/t$. Nontrivial behavior results when the model is formally continued below two dimensions. Although the physical interpretation of the Navier-Stokes equations below $d = 2$ is unclear, the results apply to a forced Burger's equation in one dimension. A large class of models produces a spectral function $E(k)$ which behaves as k^2 in three dimensions, as expected on the basis of equipartition. However, nonlinear effects (which become significant below *four* dimensions) control the infrared properties of models which force the Navier-Stokes equations at zero wave number.

I. INTRODUCTION

Renormalization group methods have enjoyed success in fields as disparate as high-energy physics,[1] critical phenomena,[2] and solid-state physics.[3] In particular there has been considerable progress in the application of renormalization group theory to study *dynamic* critical phenomena.[4] The Navier-Stokes equation for an incompressible fluid with a random forcing function bears a superficial resemblance to various models used in studies of nonlinear spin dynamics.[4,5] Here we exploit this similarity to give an analysis of the large-distance, long-time behavior of velocity correlations generated by the Navier-Stokes equations with a variety of different forcing functions. Although our analysis does *not* pertain to the properties of a fluid near its critical point,[6] renormalization group methods useful in the study of critical dynamics can be taken over directly.

In Sec. II we will discuss different possibilities for the behavior of the force-force correlations at small wave number. One of the models (model A), with the force regarded as a noise field simulating the effects of the molecular degrees of freedom, describes a fluid near thermal equilibrium. This model generates the familiar long-time tails in the renormalized viscosity, and produces new singularities at small wave numbers as well. Renormalization group theory leads naturally to a unified treatment of these anomalies and provides a scaling description of the breakdown of hydrodynamics which occurs for $d<2$.

Other models are also described in Sec. II. A kind of universality applies: large classes of models exhibit similar infrared, long-time properties. In particular, the more "realistic" models all exhibit a spectral density function $E(k)$ which goes as k^{d-1} at small wave number. This agrees with a result obtained by Saffman[7] for homogeneous isotropic turbulence. We should note, of course, that our considerations refer to the region of effectively small Reynolds' number, and $E(k)\sim k^{d-1}$ is simply a consequence of equipartition and phase-space considerations. It is all the more noteworthy, therefore, that one model (model B), which forces the Navier-Stokes equation even at $k=0$, leads to rather different results at small k. Here nonlinearities dominate the infrared behavior of $E(k)$ below four dimensions, and lead to logarithmic anomalies in $d=4$.

We have supplemented our analysis of the Navier-Stokes equation with a brief discussion of two additional equations appropriate to fluid behavior: A forced version of Burger's equation in one dimension, and a model of the diffusion of a passive scalar. The results for Burger's equation appear to be new, and it would be interesting to test them in a numerical simulation.

In Sec. II we define and discuss the model systems considered in this paper. We discuss the relationship of our analysis to previous work on

long-time tails, as well as the molecular basis for the continuum equations employed here. In Sec. III we present a recursion relation analysis of the large-distance, long-time properties of a variety of fluid systems. Although the calculations will be presented in some detail, reference will be made to the exposition of the dynamic renormalization group approach given for ferromagnetic systems by Ma and Mazenko.[8]

It will be shown in Sec. IV that many of the results of Sec. III can be derived quickly and efficiently by a direct graphical approach. The graphical analysis is simplified considerably by a Ward identity related to the Galilean invariance of the underlying dynamical equations. This same identity shows that certain results obtained in Sec. III are, in fact, valid to all orders in ϵ (here, ϵ is either $2-d$ or $4-d$). The graphical manipulations are conveniently performed using the formalism developed by Martin, Siggia, and Rose.[9] We will not derive this formalism here but note that it can be quickly developed from the equations of motion described in Sec. III using a path integral approach.[10] Although the graphical analysis quickly produces results derived somewhat more laboriously in Sec. III the recursion relation method appears to be more generally applicable, and should be of more utility when Ward identities do not produce such enormous simplifications.

Section V summarizes what has been accomplished.

II. DYNAMICAL EQUATIONS

A. The forced Navier-Stokes equation

Consider the Navier-Stokes equation describing an unbounded incompressible fluid subject to a random forcing function $\vec{f}(\vec{x},t)$, namely,

$$\partial_t\vec{v}+\lambda_0(\vec{v}\cdot\vec{\nabla})\vec{v}=-\lambda_0\vec{\nabla}p/\rho+\nu_0\nabla^2\vec{v}+\vec{f}\,, \tag{2.1}$$

where $\vec{v}=\vec{v}(\vec{x},t)$ is the fluid velocity, $p=p(\vec{x},t)$ is the pressure, ρ is the fluid mass density, ν_0 is the (unrenormalized) viscosity, and λ_0 is a perturbative parameter which will eventually be set to unity. The pressure term in (2.1) is used to enforce the condition of incompressibility,

$$\vec{\nabla}\cdot\vec{v}=0\,. \tag{2.2}$$

Equation (2.1) has often been considered as a model of homogeneous, isotropic turbulence.[11] The use of a statistically defined force sidesteps consideration of the onset of turbulence with increasing Reynolds number, and allows the generation of the statistically steady state. If we take the force to act only at long wavelengths, it is plausible that it sets up an inertial range cascade which, at short distances, is independent of the details of the force.[12]

We are concerned here with the *infrared, long-time* properties of correlations generated by $\vec{f}(\vec{x},t)$. Clearly, the long-wavelength fluid behavior will depend to some extent on the properties of the force. However, as mentioned in Sec. I, some degree of universality applies even at large distances. While we will not attempt to treat the formidable and probably more interesting problem of the ultraviolet (short-distance, short-time) correlations described by (2.1), i.e., of fully developed turbulence, the infrared properties are interesting in their own right. Batchelor and Proudman[13] originally considered the behavior of the large eddies in freely decaying (unforced) turbulence. Specifically, they studied the residual motion far downstream in a wind tunnel experiment, on a scale which is larger than the mesh which initially produced the turbulence. (In this region most of the energy in the fluid has been dissipated, and the Reynolds number is effectively small.) Saffman[7] has considered turbulence generated at an initial instant by a distribution of random impulsive forces, and finds that the spectral energy density behaves as k^{d-1} for small k. Although forced turbulence is a somewhat different problem than that treated by the above authors, we do find that most of the models considered in this section behave as predicted by Saffman. We consider, in addition, the low-frequency properties of the correlations, and corrections to the leading behavior of $E(k)$. Large-distance, long-time properties of freely decaying turbulence can be treated by the methods described in Sec. III, but will not be discussed further in this paper.

We now specify the statistical properties of the force entering Eq. (2.1). The force is taken to be purely solenoidal without loss of generality — any longitudinal component can be absorbed into the definition of the pressure. The problem is simplified further by assuming Gaussian "white noise" statistics for the force. Deviations from a strictly Gaussian force can be considered, but these do not alter the asymptotic infrared behavior. Thus, it is only necessary to specify the two-point force correlations which are of the form

$$\langle f_i(\vec{k},\omega)f_j(\vec{k}',\omega')\rangle = 2D(k)(2\pi)^{d+1}\delta(\vec{k}+\vec{k}')\delta(\omega+\omega')(\delta_{ij}-k_ik_j/k^2)\,, \tag{2.3}$$

where $\vec{f}(\vec{k},\omega)$ is the Fourier transform of $\vec{f}(\vec{x},t)$ in space and time,

$$\vec{f}(\vec{k},\omega)=\int dx\int dt\,\vec{f}(\vec{x},t)e^{i\omega t-i\vec{k}\cdot\vec{x}}\,. \tag{2.4}$$

Three prototype models will be discussed cor-

responding to different forcing functions $D(k)$:

$$\text{model A: } D(k) = D_0 k^2 \quad |\vec{k}| < \Lambda, \quad = 0, \text{ otherwise.} \tag{2.5}$$

$$\text{model B: } D(k) = D_0 \quad |\vec{k}| < \Lambda, \quad = 0, \text{ otherwise.} \tag{2.6}$$

$$\text{model C: } D(k) = D_0 \quad \tilde{\Lambda} < |\vec{k}| < \Lambda, \quad = 0, \text{ otherwise.} \tag{2.7}$$

Model A[14] can be considered simply as a Langevin model for a fluid near equilibrium. This connection will be discussed further in Sec. IIC. In this case the fluctuation-dissipation theorem requires $D_0 = \nu_0 k_B T/\rho$. It can also be thought of as representing some macroscopic stirring force whose spatial integral vanishes. Model B includes a statistically defined force which acts on the fluid even at $k = 0$. Heuristically, it corresponds to a macroscopic "shaking" of the fluid container.[15] While it is perhaps somewhat artificial to imagine exciting a fluid even at $k = 0$, model B does exhibit intriguing behavior below four dimensions. Model C, with $0 < \tilde{\Lambda} < \Lambda$, is perhaps the most realistic. The fluid is excited in a band in k space below $k = \Lambda$, and one is interested in the resulting correlations near $k = 0$. We shall show that the infrared behavior of model C is the same as that of model A, which is a further motivation for considering model A. Of course, we can consider variations, such as an $O(k^4)$ correction to $D(k)$ in model A. However, model A and model B will turn out to be representative of two broad universality classes; most variations turn out to be irrelevant variables in the sense of Wilson.[16]

It should be emphasized that the cutoff Λ occurring in Eqs. (5)–(7) has here the interpretation of an inverse stirring length (except in the case of a fluid near thermal equilibrium, see Sec. IIC). It will be in general much less than the inverse of any dissipation length scale (we will focus primarily on the limit of small viscosity) and of course much less than any molecular cutoff.

B. Burger's equation and the diffusion of a passive scalar

In Sec. III it will be shown that model A develops nontrivial behavior when formally continued below two dimensions. However, an incompressible fluid is not of much interest in, say, one dimension where the correlations vanish identically! Moreover, recent work by Frisch, Lesieur, and Sulem[17] suggests that Navier-Stokes turbulence may not be realizable for *any* dimension less than two.

Fortunately, the same renormalization group methods apply to a d-dimensional generalization of Burger's equation.[18] We consider a velocity field which, at $d = 3$, obeys

$$\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \nu_0 \nabla^2 \vec{v} + \vec{f}, \tag{2.8}$$

with the restriction

$$\vec{\nabla} \times \vec{v} = 0. \tag{2.9}$$

Using the identity $(\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{1}{2}\vec{\nabla} v^2 - \vec{v} \times (\vec{\nabla} \times \vec{v})$ and deleting the last term, this model is trivially continued to arbitrary d. In one dimension, the ultraviolet behavior is of interest in its own right, and is dominated by shock-wave excitations.[18] We will exhibit new singularities in the one-dimensional large-distance, long-time properties of this model for a Gaussian random stirring force with correlation

$$\langle f_i(\vec{k}, \omega) f_j(\vec{k}', \omega') \rangle = 2D_0 k_i k_j (2\pi)^{d+1} \delta(\vec{k} + \vec{k}') \delta(\omega + \omega'). \tag{2.10}$$

We have also considered a model of the diffusion of a passive scalar T which satisfies an equation of motion

$$\partial_t T + (\vec{v}\cdot\vec{\nabla})T = \kappa_0 \nabla^2 T. \tag{2.11}$$

Here $\vec{v}$ is the fluctuating velocity field appropriate to model A. This problem has also been treated in the long-time tail literature.[19]

C. Long-time tails and the d-dimensional Navier-Stokes equation

The problem of long-time tails in Green-Kubo functions is physically very different from those introduced above; it concerns the dynamical properties of a fluid near thermal equilibrium. More precisely, it concerns the question of whether the usual *linearized* Navier-Stokes equations correctly describe the space and time correlations of spontaneous velocity fluctuations (or, equivalently, the relaxation to equilibrium of sufficiently small externally induced fluctuations), at least for large distances and long times. It is, of course, always formally possible to reorganize the Liouville equation for the local microscopic velocity into the suggestive form[20]

$$\partial_t \vec{v}(\vec{x}, t) - \int_0^t dt' \int d\vec{x}'\, \nu_R(\vec{x} - \vec{x}', t - t') \nabla'^2 \vec{v}(\vec{x}', t') = \vec{f}_R(\vec{x}, t), \tag{2.12}$$

where the "random force" $\vec{f}_R$ vanishes on the average, is uncorrelated with the initial value $v(\vec{x}, 0)$, and where

$$\langle f_{Ri}(\vec{x}, t) f_{Rj}(\vec{x}', t') \rangle = -(k_B T/\rho) \nabla^2 \nu_R(x - x', t - t') \delta_{ij} \tag{2.13}$$

because of equipartition and the fluctuation-dissipation theorem. Here the angular brackets signify the usual thermal equilibrium average, of

course, and we denote here by $\vec{v}$ the transverse part of the unaveraged fluid velocity so that $\vec{\nabla}\cdot\vec{v}=0$. Equation (2.12) is then, of course, equivalent to the customary definition[21] of a renormalized viscosity in terms of the velocity-velocity correlation function. Conventional hydrodynamics results if one assumes that all of the many modes which contribute to $\vec{f}_R$ decay over microscopic distances and times. Then the spatial Fourier transform of ν_R is effectively of the form

$$\nu_R(k,t-t')=2\nu\delta(t-t') \tag{2.14}$$

for sufficiently small k and long times, where ν is the measured viscosity.

The long-time tails are corrections to (2.14) of the form $\sim(t-t')^{-d/2}$ in $d>2$ dimensions. They must result from microscopically slow modes which are still contained in $\vec{f}_R$. In accord with previous work[22] we make the plausible assumption that the only such terms are products of the conserved amplitudes which, for sufficiently small wave vectors, certainly decay over macroscopic times as well. (In order to exclude for simplicity terms which involve coupling of sound and heat diffusion modes we restrict consideration to an incompressible isothermal fluid.) A kinetic equation in which such mode coupling terms are still explicitly exhibited should be local and Markovian. However, it can have those properties only for wave numbers k below a cutoff Λ where Λ^{-1} is large on a microscopic scale but small on a macroscopic one. The former restriction is necessary, of course, for any continuum theory; small wavelength velocity fluctuations are, instead, treated as noise, along with all other molecular degrees of freedom. Note that for sufficiently large Λ^{-1} the relative importance of highly nonlinear terms is strongly reduced from phase-space considerations which our renormalization-group analysis renders more precise. The second restriction must be imposed, of course, to guarantee that *only* small wavelength fluctuations are treated as noise.

There is no difficulty in formally "deriving" the corresponding kinetic equation by way of projector techniques.[23] Using the standard Zwanzig-Mori procedure, one would project the Liouville dynamics onto that part of phase space which is spanned by, in principle, arbitrary powers of the velocity variable, with wave number k small than Λ, and treat the remainder as stochastic noise. In fact the resulting equation is strongly restricted by the requirements of momentum conservation, symmetry, incompressibility, and Galilean invariance. If we omit terms of higher than second order in v, and keep only the lowest-order terms in a gradient expansion, we obtain model A, i.e.,

$$(\partial_t-\nu_0\nabla^2)\vec{v}+(\vec{v}\cdot\vec{\nabla})\vec{v}=\vec{f}_0\,, \tag{2.15}$$

where because of equipartition and the fluctuation-dissipation theorem the two-point force correlations are given by (2.3), with

$$D(k)=\nu_0(k_BT/\rho)k^2\,. \tag{2.16}$$

Note that ν_0 is now a "bare" viscosity which does not as yet contain the contribution made by the mode coupling term. Since we further know, from statistical mechanics, that the equilibrium distribution of $\vec{v}$ is Gaussian, it is consistent to choose the noise force f_0 Gaussian as well.

This is the model which Zwanzig[22] has solved to second order in perturbation theory. For dimension $d\geq 3$, the result,

$$\nu_R(k=0,t)=(k_BT/\rho)\left[(d^2-2)/(d^2+2d)\right](8\pi\nu t)^{-d/2} \tag{2.17}$$

for large t, is fairly convincing since perturbation theory converges term by term. However, as Zwanzig noted, at $d=2$ perturbation theory is questionable since it diverges logarithmically, term by term, at low frequency and wave number. (The expansion parameter is $\bar{\lambda}^2=(k_BT/\rho\nu_0^2)$ whose "naive" dimension is $d-2$.) The renormalization group methods which we discuss below overcome this difficulty. Further, on the basis of Wilson's[16] ideas about irrelevant variables, they afford a convincing and simple demonstration that results obtained from model A will not be invalidated if one extends the model in several respects: to include higher derivative terms than those exhibited in (2.15), terms of higher order in $\vec{v}$, accompanying non-Gaussian contributions to the noise, or even velocity-dependent noise forces. For a near-equilibrium fluid, this is particularly important since model A can only be a crude approximation of the problem which one actually would like to solve. Of course, even renormalization-group methods do not allow one to escape entirely from the assumption, physically plausible as it may be, on which our considerations are based: That there is a coarse-grained level of description at which a kinetic equation, roughly of the structure studied here, describes the dynamics of an incompressible fluid properly.

We conclude this section with a brief note on Galilean invariance since it plays a major simplifying role in this work. Obviously, the transformation

$$\vec{v}(\vec{x},t)\to\vec{v}_0+\vec{v}'(\vec{x}-\vec{v}_0t,t)\,, \tag{2.18}$$

which is an exact symmetry of the microscopic dynamics, must also be a symmetry of semi-phenomenological models. It is important, in our case, because it *prescribes* the vertex of the nonlinearity in Eq. (2.15). [If one introduces a book-

keeping parameter λ_0 as in Eq. (2.1), it multiplies only the $\vec{v}_0$ in the argument of (2.18).] Of course, in formal "derivations" of kinetic models via projector methods, one must *require* that the "noise" term be separately invariant since otherwise an interpretation as a Langevin force would obviously be nonsensical. The standard Zwanzig-Mori technique is in accord with this requirement.

III. RECURSION RELATION ANALYSIS

A. Renormalization-group method

It is useful at this point to review the dynamic renormalization-group procedure.[4] Corresponding to the cutoff Λ in the definitions of the forcing function, the Fourier decomposition of the velocity field will be cut off for $|\vec{k}| > \Lambda$:

$$v_i(\vec{x}, t) = \int_{k<\Lambda} \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi} v_i(\vec{k}, \omega) e^{i\vec{k}\cdot\vec{x} - i\omega t} . \quad (3.1)$$

In reality, of course, the high-k velocity modes will eventually be excited[24] by the nonlinearities in the Navier-Stokes equation even for a cutoff force. The supposition here is that the ultraviolet excitations do not affect the infrared modes populating the steady state which develops for $k \ll \Lambda$. Indeed, in line with the usual arguments advanced for the universality of critical phenomena,[2] we expect that the short-distance properties are irrelevant to the large-distance, long-time behavior. Short-distance phenomena will, of course, affect the *amplitudes* of power-law singularities in the correlation functions. These amplitudes are nonuniversal, and cannot be determined by the renormalization techniques described here.

Substituting the decomposition (3.1) into (2.1), and using (2.2) to eliminate the pressure, we obtain the transformed Navier-Stokes equation,

$$v_l(\vec{k}, \omega) = G_0(k, \omega) f_l(\vec{k}, \omega) - \tfrac{1}{2} i \lambda_0 G_0(k, \omega) P_{lmn}(\vec{k}) \times \int_{q\Omega} v_m(\vec{q}, \Omega) v_n(\vec{k} - \vec{q}, \omega - \Omega) . \quad (3.2)$$

Here we have defined an unrenormalized propagator,

$$G_0(k, \omega) \equiv [-i\omega + \nu_0 k^2]^{-1} , \quad (3.3)$$

and the function

$$P_{lmn}(\vec{k}) \equiv P_{lm}(\vec{k}) k_n + P_{ln}(\vec{k}) k_m , \quad (3.4)$$

where $P_{ij}(\vec{k})$ is the transverse projection operator,

$$P_{ij}(\vec{k}) \equiv \delta_{ij} - k_i k_j / k^2 . \quad (3.5)$$

Furthermore, we have adopted a standard convention by defining

$$\int_{q\Omega} \equiv \int_{q<\Lambda} \frac{d^d q}{(2\pi)^d} \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} . \quad (3.6)$$

A particular mode $v_l(\vec{k}, \omega)$ is coupled to the remaining degrees of freedom by the nonlinear term proportional to λ_0 on the right-hand side of (3.2). In principle Eq. (3.2) can be iterated in powers of this coupling.

In practice, however, there are difficulties. Consider model A for concreteness. Note that expressing time in units of $1/\nu_0$, velocity in units of $(D_0/\nu_0)^{1/2}$ and the force in units of $1/(D_0\nu_0)^{1/2}$ amounts to setting $D_0 = \nu_0 = 1$ and replacing the vertex λ_0 by

$$\bar{\lambda}_0 = \lambda_0 D_0^{1/2} / \nu_0^{3/2} \quad (3.7)$$

Thus while one naively calculates a perturbation expansion in powers of λ_0 the actual series involves powers of $\bar{\lambda}_0$. This is fortunate since λ_0 was only introduced to organize the expansion, and must eventually be set to unity. However, there are obvious difficulties with a naive perturbation theory for small viscosities ν_0. Note further that $\bar{\lambda}_0$ has the dimension of length to the power $\frac{1}{2}(d-2)$. Thus an expansion of the dimensionless ratio ν_R/ν_0, for example, where ν_R is the renormalized viscosity at zero frequency and wave number, must involve terms of the form

$$\nu_R/\nu_0 = 1 + \text{const}\ \bar{\lambda}_0^2 \int_q (1/q^2) + \cdots , \quad (3.8)$$

which, in two dimensions, diverge logarithmically due to small wave-vector fluctuations.

It should be unproblematic, however, to selectively assess the effect which modes in a shell $\Lambda e^{-l} < |\vec{k}| < \Lambda$ have on the dynamics of the remaining ones if l is not chosen too large. In effect this is what the renormalization group does. One projects the equations of motion onto the phase space spanned by modes with $0 < |\vec{k}| < \Lambda e^{-l}$, and pushes the remainder into the appropriately redefined, partially renormalized noise. This is clearly an iterative procedure from whose asymptotic behavior the properties of macroscopic modes can be extracted.

As explained by Ma and Mazenko,[8] the dynamic renormalization group procedure consists of two steps. First, we eliminate from (3.2) the modes $v_i^>(k, \omega)$ such that $\Lambda e^{-l} < |\vec{k}| < \Lambda$. This is done by formally solving the equations for $v_i^>(k, \omega)$ as a power series in λ_0. The solution, because of the nonlinearities, depends on the remaining modes $v_i^<(k, \omega)$. These formal solutions are then substituted into the equations for $v_i^<(\vec{k}, \omega)$ to eliminate their explicit dependence on $v_i^>(\vec{k}, \omega)$. Finally, the reduced set of equations is averaged over the part of the force $f_i^>(\vec{k}, \omega)$ that acts in the shell $\Lambda e^{-l} < |\vec{k}| < \Lambda$.[25] This redefines the coefficients which enter the reduced equations of motion. The fluctuating remainder is added to the noise force $f_i^<(\vec{k}, \omega)$,

and redefines the coefficients which characterize its spectrum. The parameter l is a measure of the fraction of the degrees of freedom which have been eliminated.

The second step consists of rescaling space, time, and the remaining velocities and forces in order to make the new set of equations look as much as possible like the original Navier-Stokes equation. For example, we would like to keep the coefficient of $\partial_t \vec{v}$ in Eq. (2.1) fixed at unity.

This procedure will in general produce complicated new couplings (higher powers of $\vec{v}$ and $\vec{\nabla}$), in addition to the original Navier-Stokes nonlinearities. However, these couplings turn out to be irrelevant variables, and can often be neglected to a leading approximation (for example, they can be neglected in model A above two dimensions, and also to first order in $\epsilon = 2 - d$ below two dimensions). The direct graphical approach discussed in Sec. IV bypasses the complications introduced by these new couplings.

The output of almost any renormalization group calculation can be expressed in terms of a homogeneity relation. Consider for concreteness the velocity-velocity correlation function,

$$G_{ij}(\vec{k}, \omega) = \frac{\langle v_i(\vec{k}, \omega) v_j(\vec{k}', \omega') \rangle}{(2\pi)^{d+1} \delta(\vec{k}+\vec{k}') \delta(\omega+\omega')} \tag{3.8}$$

for model A. We will bypass the problems which arise in a direct expansion in powers of $\bar{\lambda}_0$ by means of a scaling law (to be derived in Sec. III B), namely

$$G_{ij}(\vec{k}, \omega; \lambda_0) = e^{\alpha(l)} G_{ij}[e^l \vec{k}, e^{\alpha(l)}\omega; \bar{\lambda}(l)]. \tag{3.9}$$

This scaling or homogeneity relation holds for small k and ω. It expresses the fact that G_{ij} can be computed from both the original and the reduced set of equations. Here $\alpha(l)$ is a function characterizing the time rescaling necessary to preserve the form of the Navier-Stokes equation, and $\bar{\lambda}(l)$ is the effective coupling constant after a fraction $1 - e^{-dl}$ of the degrees of freedom have been eliminated.

The left-hand side of (3.9) is difficult to calculate if $\bar{\lambda}_0$ is large and/or, in two dimensions, k and ω are small. In fact we will discover that, for $d \geq 2$, $\bar{\lambda}(l)$ can be made as small as desired by choosing l sufficiently large. Since $\alpha(l)$ turns out to be a linearly increasing function of l, we can, for large l, compute the right-hand side of (3.9) by ordinary perturbation theory. The assertion that $\bar{\lambda}(l) \to 0$ as $l \to \infty$ might be called "infrared asymptotic freedom" in the language of quantum field theory.[1]

B. Model A

We now carry out the program outlined above for model A. The calculations are very similar to a calculation described in Sec. IV B of the paper by Ma and Mazenko.[8] These authors consider a different problem, but one which also involves a quadratic nonlinearity. The recursion relations are conveniently expressed in terms of Feynman graphs. The graphs which occur are the same as those appearing in the standard graphical expansions of the Navier-Stokes equation,[26] but with the interpretation that the internal lines carry momenta in the range $e^{-l}\Lambda < |\vec{k}| < \Lambda$ and the external lines carry momenta such that $|\vec{k}| < e^{-l}\Lambda$. Both internal and external frequencies are unbounded.

Intermediate recursion relations for the propagator, the force-force correlation, and for λ are shown schematically in Fig. 1 to leading order in $\bar{\lambda}$. The term "intermediate" is used because we have not yet rescaled space, time, etc. From Fig. 1, one can simply "read off" the intermediate recursion relations for ν, D, and λ. As an illustration, we evaluate the recursion formula implied by Fig. 1(a) in Appendix A. The reader is referred to Ref. 8 for more details.

The intermediate values of ν, D, and λ are

$$\nu_I = \nu_0[1 + A_d \bar{\lambda}_0^2 (1 - e^{-(d-2)l})/(d-2)], \tag{3.10}$$

$$D_I = D_0[1 + A_d \bar{\lambda}_0^2 (1 - e^{-(d-2)l})/(d-2)], \tag{3.11}$$

$$\lambda_I = \lambda_0. \tag{3.12}$$

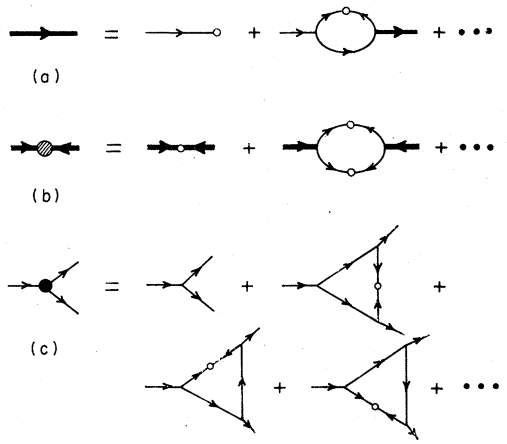

FIG. 1. Schematic representation of recursion formulas describing propagator, correlation function, and vertex renormalization to leading order for the Navier-Stokes equations. Intermediate frequencies are summed from $-\infty$ to $+\infty$, while the intermediate momenta $\vec{q}$ are integrated over the shell $e^{-l}\Lambda < |\vec{q}| < \Lambda$. A light line terminating in an open circle represents the bare propagator $G_0(k, \omega)$ times the random force. Heavy lines represent the full solution of the interacting Navier-Stokes equation, $v_I(\vec{k}, \omega)$. Figure 1(a) describes viscosity renormalization, Fig. 1(b) describes renormalization of the force-force correlations, and Fig. 1(c) gives the renormalization of the coupling λ_0. See Fig. 3 and Appendix A for more details.

Here A_d is a dimensionality-dependent constant, positive for $d \geq 2$ and equal to $1/16\pi$ in two dimensions, and we have, for convenience, set $\Lambda = 1$. As in the model considered in Ref. 8, the "vertex correction" contributions to λ_I shown in Fig. 1(c) vanish identically. These intermediate couplings enter an intermediate Navier-Stokes equation for $v_i^<(\vec{k}, \omega)$, which is of the form (3.2) but with ν_I replacing ν_0, λ_I replacing λ_0, and with $f_i^<(\vec{k}, \omega) \to f_{I,i}^<(\vec{k}, \omega)$ where

$$\langle f_{I,i}^<(\vec{k}, \omega) f_{I,i}^<(\vec{k}', \omega') \rangle = 2D_I k^2 (2\pi)^{d+1} \delta(k + k') \delta(\omega + \omega') P_{ij}(\vec{k}). \quad (3.13)$$

To implement the second step of the dynamic renormalization group procedure we set

$$\vec{v}^<(\vec{k}, \omega) = \zeta(l) \vec{v}'(\vec{k}', \omega'), \quad (3.14)$$

where

$$\vec{k}' = e^l \vec{k} \text{ and } \omega' = e^{\alpha(l)} \omega. \quad (3.15)$$

The rescaling of k compensates for the eliminated degrees of freedom, and $\zeta(l)$ and $\alpha(l)$ are to be determined. The force must be rescaled by

$$\vec{f}_I^<(\vec{k}, \omega) = e^{-\alpha(l)} \zeta(l) \vec{f}'(\vec{k}', \omega'). \quad (3.16)$$

The recursion relations for ν and D are then given by

$$\nu' \equiv \nu(l) = e^{\alpha(l) - 2l} \nu_I(l), \quad (3.17)$$

$$D' \equiv D(l) = e^{\alpha(l) - 2l} [\exp[\alpha(l) + \tfrac{1}{2} dl] / \zeta(l)]^2 D_I(l). \quad (3.18)$$

We note from (3.11) and (3.12) that ν and D are renormalized in the same way. This property, which persists to all orders in perturbation theory, is maintained by the rescaling if we choose

$$\zeta(l) = \exp[\alpha(l) + \tfrac{1}{2} dl]. \quad (3.19)$$

With this choice the recursion formula for λ is

$$\lambda' \equiv \lambda(l) = \exp[\alpha(l) - \tfrac{1}{2}(d+2) l] \lambda_I. \quad (3.20)$$

Because the renormalization group can obviously be iterated it is convenient to replace l by an infinitesimal parameter δ, which leads to *differential* recursion relations. More accurate values of the parameters $\nu(l)$, $D(l)$, and $\lambda(l)$ (which describe the system after a *finite* fraction $1 - e^{-dl}$ of the degrees of freedom have been eliminated) are then obtained by integrating these differential equations. On taking the limit $\delta \to 0$ we find[14]

$$d\nu(l)/dl = \nu(l)[-2 + z(l) + A_d \bar{\lambda}^2(l)], \quad (3.21)$$

$$dD(l)/dl = D(l)[-2 + z(l) + A_d \bar{\lambda}^2(l)], \quad (3.22)$$

$$d\lambda(l)/dl = \lambda(l)[-1 - \tfrac{1}{2} d + z(l)], \quad (3.23)$$

where

$$\alpha(l) = \int_0^l z(l') dl' \quad (3.24)$$

and

$$\bar{\lambda}(l) = \lambda(l)[D(l)/\nu^3(l)]^{1/2} \quad (3.25)$$

in accord with (3.7).

The function $z(l)$ is still arbitrary at this point. However, $z(l)$ drops out of the recursion relation for the reduced coupling $\bar{\lambda}$, namely,

$$d\bar{\lambda}(l)/dl = \tfrac{1}{2} \epsilon \bar{\lambda}(l) - A_d \bar{\lambda}^3(l), \quad (3.26)$$

where we have set

$$\epsilon \equiv 2 - d. \quad (3.27)$$

Above two dimensions ($\epsilon < 0$), the recursion formula (3.26) drives $\bar{\lambda}(l)$ to zero as $l \to \infty$. In exactly two dimensions, $\bar{\lambda}(l)$ still goes to zero, although much more slowly. Below $d = 2$, the fixed point at $\bar{\lambda} = 0$ is unstable to small perturbations, and $\bar{\lambda}(l)$ (for $\bar{\lambda}_0 > 0$) is driven to a stable fixed point,

$$\bar{\lambda}^* = (8\pi\epsilon)^{1/2}, \quad (3.28)$$

to order $\epsilon^{3/2}$. The solution of (3.25), which displays the behavior described above, is

$$\bar{\lambda}(l) = \bar{\lambda}_0 \exp \tfrac{1}{2} \epsilon l [1 + 2A_d \bar{\lambda}_0^2 (e^{\epsilon l} - 1)/\epsilon]^{1/2}. \quad (3.29)$$

The scaling relation to be derived below simplifies considerably if $z(l)$ is chosen to keep $\nu(l)$ and thus $D(l)$ fixed at their initial values. From (3.22) and (3.23) we see that the necessary, l-dependent choice is

$$z(l) = 2 - A_d \bar{\lambda}^2(l). \quad (3.30)$$

Thus, as $l \to \infty$, $z(l)$ approaches a fixed value,

$$\begin{aligned} z &\to 2, \quad d \geq 2, \\ z &\to 2 - \tfrac{1}{2}\epsilon, \quad d < 2. \end{aligned} \quad (3.31)$$

The homogeneity relation (3.9) follows from the fact that for $k < e^{-l}$, $G_{ij}(\vec{k}, \omega)$ can be computed both from the original and the reduced set of equations. Because of the velocity scaling (3.14) we have

$$G_{ij}(\vec{k}, \omega; \bar{\lambda}_0) = \frac{\langle v_i(\vec{k}, \omega) v_j(\vec{k}', \omega') \rangle}{(2\pi)^{d+1} \delta(\vec{k} + \vec{k}') \delta(\omega + \omega')} = \zeta^2(l) e^{-dl + \alpha(l)} \frac{\langle v_i^<(e^l \vec{k}, e^{\alpha(l)} \omega) v_j^<(e^l \vec{k}', e^{\alpha(l)} \omega') \rangle}{(2\pi)^{d+1} \delta(e^l \vec{k} + e^l \vec{k}') \delta(e^{\alpha(l)} \omega + e^{\alpha(l)} \omega')}$$

$$= \exp\left(\int_0^l z(l')\, dl' \right) G_{ij}\left[e^l \vec{k}, \exp\left(\int_0^l z(l')\, dl' \right) \omega; \bar{\lambda}(l) \right]. \quad (3.32)$$

Above two dimensions, $\int_0^l z(l')\,dl'$ approaches 2 for large l, and $\bar{\lambda}(l)$ tends to zero. In this case Eq. (3.32) implies that, for asymptotically small k, $G_{ij}(\vec{k},\omega)$ can be expressed in term of a scaling function,[2]

$$G_{ij}(\vec{k},\omega)=P_{ij}(\vec{k})k^{-2}\Phi(\omega/k^2). \tag{3.33}$$

This is just the prediction of conventional linearized hydrodynamics, provided we take

$$\Phi(x)=\chi\nu/[x^2+\nu^2], \tag{3.34}$$

where $\chi\equiv D_0/\nu_0=k_BT/\rho$, and ν is the measured viscosity.

Indeed, the renormalization (in the conventional sense) of this transport coefficient is easily extracted from (3.32) as well since for sufficiently large l [such that $\bar{\lambda}(l)\simeq 0$] we have more accurately

$$\int_0^l z(l')\,dl'-2l\equiv-\Delta=-A_d\int_0^\infty \bar{\lambda}(l')\,dl', \tag{3.35}$$

except for exponentially small terms. Since we know the correlation function for $\lambda\simeq 0$, we obtain (3.34), with

$$\nu/\nu_0=e^{\Delta}=[1+2A_d\bar{\lambda}_0^2\Lambda^{|\epsilon|}/|\epsilon|]^{1/2}, \tag{3.36}$$

where we have restored the wave-number cutoff Λ. In an equilibrium fluid above two dimensions, this formula gives the contribution of modes with wave number below Λ to the measured viscosity.

In order to discuss corrections to conventional hydrodynamics it is convenient to consider the renormalized viscosity $\nu_R(k,\omega)$, defined implicitly by representing $G_{ij}(\vec{k},\omega)$ in the form[21]

$$G_{ij}(\vec{k},\omega)=2P_{ij}(\vec{k})\,\mathrm{Re}\{\chi/[-i\omega+k^2\nu_R(k,\omega)]\}. \tag{3.37}$$

This definition agrees with that implied by Eq. (2.12). A scaling relation for $\nu_R(k,\omega)$ follows, namely

$$\nu_R(k,\omega;\bar{\lambda}_0)=e^{2l-\alpha(l)}\nu_R[e^lk,e^{\alpha(l)}\omega;\bar{\lambda}(l)]. \tag{3.38}$$

As an example consider the case of $k=0$. We evaluate the right-hand side of (3.38) at $l=l^*$ such that

$$e^{\alpha(l^*)}\omega=1. \tag{3.39}$$

By choosing ω small enough we can make the effective coupling, $\bar{\lambda}(l^*)\sim\bar{\lambda}_0\omega^{(1/4)|\epsilon|}$, as small as desired so that $\nu_R[0,1;\bar{\lambda}(l^*)]$ can be expanded. Noting that perturbation theory involves only even powers of $\bar{\lambda}$ it is clear that the lowest-order correction is of order $\omega^{(1/2)|\epsilon|}$. This argument, which is just a simple application of Wegner's theory[27] of the corrections to scaling, has thus produced the celebrated long-time tail correction to the renormalized viscosity. Explicitly in $d=3$ we find

$$\nu_R(0,\omega)=\nu+[7i\chi/120\pi\nu](i\omega/2\nu)^{1/2}+\mathcal{O}(\omega) \tag{3.40}$$

in agreement with the long-time tail literature.[19]

In exactly *two dimensions* the slow approach of $\bar{\lambda}(l)$ to zero gives logarithmic corrections to conventional hydrodynamics which, in contrast to the results for $d>2$, are not conveniently handled by ordinary perturbation theory but which the renormalization group method yields easily. Inserting the expressions (3.29) and (3.30) to compute $\alpha(l)$ near $d=2$ one obtains

$$\nu_R(k,\omega;\bar{\lambda}_0)=[1+\bar{\lambda}_0^2(e^{\epsilon l}-1)/8\pi\epsilon]^{1/2}\,\nu_R[e^lk,e^{z^*l}\omega;\bar{\lambda}(l)], \tag{3.41}$$

where z^* is the value given in (3.31). To find $\nu_R(k,\omega=0)$ we evaluate the right-hand side by choosing $l=l^*$ such that $e^{l^*}k=1$. We can take k small enough to make $\bar{\lambda}(l^*)$ as small as desired. Nevertheless, the wave-number argument on the right-hand side of (3.41), ke^{l^*}, remains firmly pinned at 1, and since even in two dimensions the perturbation expansion for ν_R is unproblematic for *finite* k (or ω), we can simply replace $\nu_R[1,0;\bar{\lambda}(l^*)]$ by its unrenormalized value, ν_0. The result,[14]

$$\nu_R(k,0)=\nu_0[1+\bar{\lambda}_0^2(k^{-\epsilon}-1)/8\pi\epsilon]^{1/2}, \tag{3.42}$$

is valid for $|\epsilon|\ll 1$, and gives a logarithmic correction to the hydrodynamic result in $d=2$, namely

$$\nu_R(k,0)=[(\chi/8\pi)\ln(1/k)]^{1/2} \tag{3.43}$$

for asymptotically small k. Similarly, (3.41) gives[22,14]

$$\nu_R(0,\omega)=[(\chi/16\pi)\ln(1/\omega)]^{1/2} \tag{3.44}$$

for asymptotically small frequency in two dimensions, and since $z^*=2-\frac{1}{2}\epsilon$ for $d<2$, one obtains $\nu_R\sim(1/\omega)^{\epsilon/(4-\epsilon)}$ for $d<2$.[19,14] [Equation (3.44) differs by a factor of $2^{1/2}$ from the result conventionally extracted from the mode coupling formula, see p. 117 in Pomeau and Résibois, Ref. 19. We believe that (3.44) is correct. The difference can be understood by arguing that the mode coupling formula computes not $\nu(\omega)$, but $\delta\nu(\omega)$.]

A quantity of considerable interest in turbulence theory is the spectral energy density,[12]

$$E(k)=[\tfrac{1}{2}S_d/(2\pi)^{d+1}]k^{d-1}\int_{-\infty}^{+\infty}\mathrm{Tr}G_{ij}(\vec{k},\omega)\,d\omega, \tag{3.45}$$

where S_d is the surface area of a d-dimensional sphere, $S_d=2\pi^{(1/2)d}/\Gamma(\frac{1}{2}d)$. Because the prefactor $e^{\alpha(l)}$ in (3.9) is identical to the frequency rescaling. $\int \mathrm{Tr}G_{ij}(\vec{k},\omega)\,d\omega$ is a constant for small k in any dimension, and $E(k)\sim k^{d-1}$ in accord with the results of Saffman[7] for a somewhat different problem. In fact, as shown by Edwards and McComb,[28] all equal time correlations generated by a force whose autocorrelation goes as D_0k^2 are Gaussian distributed. Anomalies (controlled by the exponent z) only ap-

pear when the correlations are considered at unequal times.

C. A generalized Burger's equation

As explained in Sec. II, there are difficulties in the physical interpretation of results obtained for model A below two dimensions. For this reason, we consider the d-dimensional Burger's equation described in Sec. II B. Although it is interesting to survey its properties as a function of dimensionality, the model is rather unrealistic outside of one dimension since it does not conserve "energy" [i.e., the integral $\int d^dx\, v^2(\vec{x}, t)$] in the inviscid limit. We shall see that there are corresponding anomalies in the recursion relation analysis near $d=2$. In this subsection we merely display these peculiarities, and demonstrate that the small viscosity infrared properties are not readily susceptible to analysis near two dimensions. Results derived in Sec. IV will, in fact, allow explicit predictions to be made about this model in one dimension.

Upon Fourier transformation, Eq. (2.8) takes the form

$$v_i(\vec{k}, \omega) = G_0(k, \omega) f_i(\vec{k}, \omega) - \tfrac{1}{2} i\lambda_0 G_0(k, \omega) k_i \int_{q\Omega} v_j(\vec{q}, \Omega) v_j(\vec{k} - \vec{q}, \omega - \Omega)\,, \tag{3.46}$$

where $G_0(k, \omega)$ is again given by (3.3), and the parameter λ_0 has been inserted multiplying the nonlinearity. The recursion analysis proceed as sketched for model A; the graphs shown in Fig. 1 again appear, but with interpretations dictated by (3.46) and (2.10). Because of Galilean invariance the "vertex correction" graphs[29] shown in Fig. 1(c) again vanish when the external momenta and frequencies are taken to zero. However, the model develops nontrivial static properties, and therefore ν_0 and D_0 no longer renormalize in the same way. We therefore rescale the velocities and forces as in (3.14) and (3.16) but extend (3.19) by means of the parametrization

$$\zeta(l) = \exp\left(\int_0^l [z(l') + y(l') + \tfrac{1}{2}d]\, dl'\right). \tag{3.47}$$

In d dimensions the recursion relations for $\nu(l)$, $D(l)$, and $\lambda(l)$ are, to leading order in $\bar{\lambda} = \lambda D^{1/2}/\nu^{3/2}$,

$$d\nu(l)/dl = \nu(l)[-2 + z(l) + K_d \bar{\lambda}^2(l)(2-d)/4d], \tag{3.48}$$

$$dD(l)/dl = D(l)[-2 + z(l) - 2y(l) + K_d \bar{\lambda}^2(l)/4], \tag{3.49}$$

$$d\lambda(l)/dl = \lambda(l)[-1 - \tfrac{1}{2}d + z(l) + y(l)\,], \tag{3.50}$$

where $K_d = 1/[2^{d-1}\pi^{(1/2)d}\Gamma(\frac{1}{2}d)]$.

As in the discussion of model A, $y(l)$ and $z(l)$ can be chosen to be functions of $\bar{\lambda}(l)$ such that $\nu(l)$ and $D(l)$ remain at their initial values. However, difficulties arise when we consider the recursion formula for $\bar{\lambda}(l)$,

$$d\bar{\lambda}(l)/dl = \tfrac{1}{2}(2-d)\bar{\lambda}(l) + K_d[(2d-3)/4d]\bar{\lambda}^3(l)\,. \tag{3.51}$$

This equation differs qualitatively from the corresponding equation for model A [Eq. (3.26)] near $d=2$ because of the sign of the cubic term. A nontrivial fixed point exists *above* two dimensions, but it is unstable to small perturbations. If $\bar{\lambda}$ is initially larger than this critical coupling, $\bar{\lambda}(l)$ becomes intractably large for large l, and homogeneity relations such as (3.9) are of little use.

It is only below 1.5 dimensions, where the cubic term changes sign, that (3.51) has any similarity to (3.26). Of course, higher-order terms in $\bar{\lambda}(l)$ may be of importance in these dimensionalities. Note that the recursion relations for $\nu(l)$ and $D(l)$ can be made identical in one dimension, where energy becomes a conserved quantity. In Sec. IV we will show that ν and D are related by a fluctuation-dissipation theorem in one dimension. This relation, together with a Ward identity related to the Galilean invariance of the theory, will lead to a nontrivial prediction in one dimension. Specifically, we will demonstrate that the correlation function,

$$G(k, \omega) = \langle v(k, \omega) v(k', \omega')\rangle / (2\pi)^2 \delta(k + k')\delta(\omega + \omega') \tag{3.52}$$

scales with an exponent $z = \frac{3}{2}$, i.e.,

$$G(k, \omega) = k^{-3/2}\Phi(\omega/k^{3/2})\,. \tag{3.53}$$

D. Convection of a passive scalar

It is instructive to study the convection of a passive scalar by the velocity field appropriate to model A. We must consider the auxiliary equation of motion (2.11), where the variable T is intended to represent temperature, or the concentration of a labelled set of particles.[11,19]

Introducing the Fourier-Laplace transform of the scalar field $T(\vec{x}, t)$,

$$\hat{T}(\vec{k}, \omega) = \int_0^\infty dt\, e^{i\omega t} \tilde{T}(\vec{k}, t) = \int_0^\infty dt \int d^dx\, e^{i\omega t - i\vec{k}\cdot\vec{x}} T(\vec{x}, t)\,, \tag{3.54}$$

Eq. (2.11) takes the form

$$\hat{T}(\vec{k}, \omega) = \mathcal{G}_0(k, \omega)\tilde{T}(\vec{k}, t=0) - i\lambda_2 \mathcal{G}_0(k, \omega) k_i \int_{q\Omega} v_i(\vec{k} - \vec{q}, \omega - \Omega)\hat{T}(\vec{q}, \Omega)\,. \tag{3.55}$$

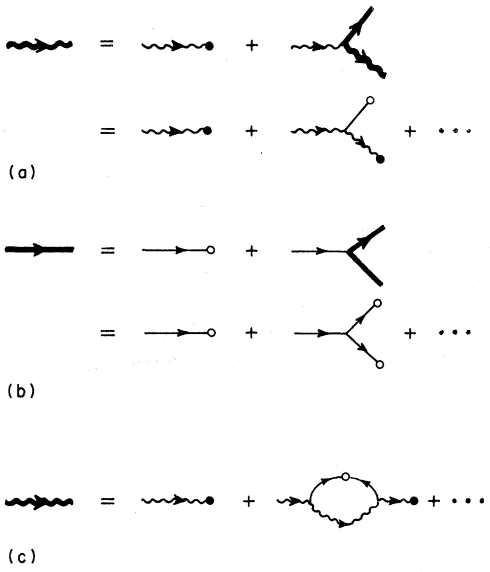

FIG. 2. Schematic representation of the coupled equations of motion describing the diffusion of a passive scalar. In Fig. 2(a), the light wavy line is the bare diffusion propagator, while the solid circle represents the initial conditions $\tilde{T}(\vec{k}, t=0)$. Heavy wavy lines indicate the solution of the diffusion equation before averaging over the random force. Figure 2(c) summarizes the recursion formula for the diffusion propagator obtained by averaging over components of the random force $f_i(\vec{k}, \omega)$ in the shell $e^{-l}\Lambda < |\vec{k}| < \Lambda$.

A parameter λ_2 has been inserted in the nonlinear term, and we have introduced the bare diffusion propagator

$$g_0(k, \omega) = [-i\omega + \kappa_0 k^2]^{-1} \ . \tag{3.56}$$

Equation (3.55), which is to be solved in conjunction with (3.2), is depicted schematically in Fig. 2(a) together with its iterated solution. Figure 2(b) shows a schematic representation of (3.2).

The relaxation from the initial condition $\tilde{T}(\vec{k}, t=0)$ results upon averaging over the fluctuating velocity field. As discussed earlier, this can be done gradually, by integrating out modes in successive shells of $\vec{k}$ space. The graphs which represent the recursion relation for the diffusion propagator are shown in Fig. 2(c). As usual, nonlinear corrections to the vertex vanish. Note that if $\lambda_2 \neq \lambda_0$, the theory is not formally Galilean invariant; even then vertex corrections vanish as a result of the transversality of the velocity field. However, since initially $\lambda_2 = \lambda_0 = 1$, Galilean invariance is assured at every recursive step if we choose the same rescaling for $\hat{T}$ as we did for the velocity field,

$$\hat{T}(\vec{k}, \omega) = \zeta(l)\hat{T}'\left\{e^{l}\vec{k}, \exp\left[\int_0^l z(l')\, dl'\right]\omega\right\}, \tag{3.57}$$

where as before

$$\zeta(l) = \exp\left[\tfrac{1}{2}dl + \int_0^l z(l')\, dl'\right]. \tag{3.58}$$

In this case $\lambda_2(l)$ and $\lambda(l)$ are identical for all l. The differential equation for $\kappa(l)$ is then

$$d\kappa(l)/dl = [-2 + z(l)]\kappa(l) + \frac{[(d-1)/d]K_d\bar{\lambda}^2(l)\nu^2(l)}{[\nu(l) + \kappa(l)]}, \tag{3.59}$$

where K_d was defined in (3.48). This equation is to be solved in conjunction with the equations (3.21)–(3.25) for model A.

After completely averaging over the fluctuating velocity field, the "temperature" relaxation is customarily described by

$$\langle \hat{T}(\vec{k}, \omega)\rangle = [-i\omega + k^2\kappa_R(k, \omega)]^{-1}\tilde{T}(\vec{k}, t=0). \tag{3.60}$$

It is then easy to show that the renormalized diffusion coefficient κ_R scales according to

$$\kappa_R(k, \omega; \bar{\kappa}_0, \bar{\lambda}_0) = e^{2l-\alpha(l)}\kappa_R[e^{l}k, e^{\alpha(l)}\omega; \bar{\kappa}(l), \bar{\lambda}(l)], \tag{3.61}$$

where the dimensionless ratio $\bar{\kappa}(l) = \kappa(l)/\nu(l)$ obeys the recursion relation

$$d\bar{\kappa}(l)/dl = -A_d\bar{\lambda}^2(l)(\bar{\kappa}(l) - 2(d-1)(d+2) \times \{(d^2-2)[\bar{\kappa}(l)+1]\}^{-1}). \tag{3.62}$$

The analysis is completed in the usual way. We can integrate the renormalization equations until $l = l^*$ such that, for example,

$$e^{2\alpha(l^*)}\omega^2 + \nu_0^2 e^{4l^*}k^4 = 1. \tag{3.64}$$

This ensures that diagrammatic corrections to (3.55) are cut off by either a finite frequency or a finite momentum so that no infrared divergences can complicate the analysis at $l = l^*$, even in two dimensions. By choosing k and ω sufficiently small, $\bar{\lambda}(l^*)$ can again be made as small as desired. Recalling that $z(l)$ was chosen to keep $\nu(l)$ fixed at ν_0, we find that $\kappa(l)$ is given by the equation

$$\frac{x^2(l)-1}{x_0^2-1}\left[\frac{[x(l)-1]/(x_0-1)}{[x(l)+1]/(x_0+1)}\right]^{\xi} = \left[1 + 2A_d\bar{\lambda}_0^2\frac{e^{\epsilon l}-1}{\epsilon}\right]^{-1}, \tag{3.65}$$

where

$$x(l) \equiv \xi[2\kappa(l)+1] \tag{3.66}$$

and

$$\xi^{-2} \equiv 1 + 8(d-1)(d+2)/(d^2-2). \tag{3.67}$$

This equation, together with (3.61) and (3.64), determines $\kappa_R(k, \omega)$ in implicit form. On setting $k=0$, it is straightforward to exhibit a long-time tail in

$\kappa_R(0,\omega)$[19] from the more general results derived here. The result in three dimensions is

$$\kappa_R(0,\omega)=\kappa+[i\chi/6\pi(\nu+\kappa)^{3/2}](i\omega)^{1/2}, \tag{3.68}$$

where κ is the measured diffusion coefficient which is given by $\kappa=\nu\kappa(l\to\infty)$. Similarly, the divergence of the super-Burnett coefficient in three dimensions is easily understood by a "corrections to scaling" analysis of $\kappa_R(k,\omega=0)$. In two dimensions the result, for asymptotically small k and ω, can be written in the intriguing form

$$\kappa_R(k,\omega)/\nu_R(k,\omega)=\bar{\kappa}(\infty)=\tfrac{1}{2}(1+\sqrt{17}). \tag{3.69}$$

This ratio remains universal if the theory is formally continued below $d=2$, except that $\bar{\kappa}(\infty)$ is replaced by the ϵ-dependent solution of $x(\infty)=1$.

We note, finally, that these results are unchanged if the scalar field $T(\vec{x},t)$ is itself driven by an independent Gaussian source of thermal noise, in addition to being convected by the velocity field. This assumption is a natural one for the case when $T(\vec{x},t)$ is to represent the density of a dilute solute.

E. Model B

The treatment of model B follows closely that sketched for model A, but with rather different results. Infrared singularities due to the nonlinearities modify the large distance properties of this model below four dimensions.

Construction of recursion relations for the equation of motion (3.2) [with a forcing function governed by (2.6)] proceeds as before. Nonlinear contributions to the renormalization of λ_0 again vanish, but D_0 and ν_0 are renormalized very differently. In particular the graph shown in Fig. 1(b) does not contribute to the constant part of the force autocorrelation, but generates instead a term proportional to k^2. This term is irrelevant when the force is rescaled to keep $D(l)$ fixed at D_0.

When velocities and forces are rescaled as indicated by Eqs. (3.14) and (3.16), we find [from (3.18) with the factor e^{-2l} removed] that the choice

$$\zeta(l)=\exp\left\{\int_0^l \tfrac{1}{2}[3z(l')+d]\,dl'\right. \tag{3.70}$$

is necessary to keep $D(l)$ fixed at D_0 and to preserve the form of the Navier-Stokes equations. The resulting recursion formulas for $\nu(l)$, $D(l)$, and $\lambda(l)$, accurate to $\Theta(\bar{\lambda}^2)$, are

$$d\nu(l)/dl=[-2+z(l)]\nu(l)+B_d\lambda^2(l)D(l)/\nu^2(l), \tag{3.71}$$

$$dD(l)/dl=0, \tag{3.72}$$

$$d\lambda(l)/dl=[-1-\tfrac{1}{2}d+\tfrac{3}{2}z(l)]\lambda(l), \tag{3.73}$$

where $B_d=K_d(d-2)/2d$ with K_d as defined in (3.48). The equation for the reduced coupling $\bar{\lambda}\equiv\lambda D^{1/2}/\nu^{3/2}$ is then

$$d\bar{\lambda}(l)/dl=\tfrac{1}{2}(4-d)\bar{\lambda}(l)-(\tfrac{3}{2})B_d\bar{\lambda}^3(l). \tag{3.74}$$

The choice of $z(l)$ which fixed $\nu(l)$ at ν_0 is

$$z(l)=2-B_d\bar{\lambda}^2(l). \tag{3.75}$$

Above four dimensions (3.64) exhibits a stable "hydrodynamic" fixed point at $\bar{\lambda}^*=0$, which leads to the conventional result $z\to 2$. Below $d=4$ however, a stable nontrivial fixed-point controls the infrared properties. Thus, the asymptotic behavior of the frequency rescaling exponent $z(l)$ is

$$\begin{aligned} &z(l)\to 2, \quad d\geq 4 \\ &z(l)\to 2-(4-d)/3, \quad d<4. \end{aligned} \tag{3.76}$$

The homogeneity law analogous to (3.32) for model B is

$$G_{ij}(\vec{k},\omega;\bar{\lambda}_0)=\exp\left[2\int_0^l z(l')\,dl'\right] \times G_{ij}\left\{e^l\vec{k},\ \exp\left[\int_0^l z(l')\,dl'\right]\omega;\bar{\lambda}(l)\right\}. \tag{3.77}$$

We will not pause to calculate the renormalized viscosity for this problem, although the methods of Secs. III B and III D can be extended straightforwardly. We note, however, that there are "long-time tails" in $\nu_R(k=0,\omega)$ above four dimensions, and logarithmic corrections in $d=4$. For example,

$$\nu_R(k,\omega=0)\sim\ln^{1/3}(1/k) \tag{3.78}$$

in four dimensions. Hydrodynamics breaks down for this model below four dimensions.

Because the prefactor in Eq. (3.77) differs from the frequency rescaling, $E(k)$ does *not* vary as k^{d-1} for this model. Indeed, using the results derived above, we readily deduce that

$$\begin{aligned} &E(k)\sim k^{d-3}, && d>4, \\ &E(k)\sim k/\ln^{1/3}(1/k), && d=4, \\ &E(k)\sim k^{1-(2/3)(4-d)}, && d<4. \end{aligned} \tag{3.79}$$

This anomalous behavior is a consequence of forcing the Navier-Stokes equation at macroscopically large wavelengths, i.e., of a nonvanishing function $D(k)$ at $k=0$.

F. Model C and universality

Model A and model B are representative of two broad universality classes; their infrared long-time properties characterize a large number of similar equations of motion. In particular, k^2 corrections to $D(k)$ for model B and k^4 corrections to $D(k)$ for model A[30] are irrelevant variables, and do not affect the asymptotic properties derived in this

section. The term irrelevant is used here in the same sense as in critical phenomena: the coefficients which parametrize such correction terms vanish at the fixed point, and their asymptotic approach is faster than that of the "leading irrelevant variable" $\bar{\lambda}$. For example, a term of the form $\mu\nabla^2 v^3(\vec{x}, t)$ in the equations of motion (because of momentum conservation and symmetry, there must be at least two gradient factors in this term) would lead to a recursion relation of the form

$$d\mu(l)/dl = [z(l) - 2 - d]\,\mu(l) + \text{diagrams}, \qquad (3.80)$$

and thus approach the fixed point as e^{-dl}, for $d \geq 2$. The exponent, $z - 2 - d$, simply characterizes the space and time dimensions of μ and is thus easily found. If, therefore, such a term were inserted in homogeneity relations like (3.38), it would only contribute a correction term of, at least, order $\omega^{(1/2)d}$ which is asymptotically negligible. In this fashion it is straightforward to demonstrate that terms of higher order in $\vec{\nabla}$, ∂_t, and $\vec{v}(\vec{x}, t)$ in the equations of motion are irrelevant, as are deviations from the Gaussian character of the noise or velocity-dependent noise forces. Of course, this analysis can only be carried out in a small neighborhood of the fixed point considered here. We have nothing to say about the mathematical possibility that a different fixed point might in fact be approached for which λ, μ etc. are finite.

Even barring this bizarre possibility one should note that the irrelevant variables, which for the equilibrium fluid are undoubtedly present in the initial model, renormalize the remaining parameters. That is, if the "time" $l = l_0$ is such that for $l > l_0$ all irrelevant parameters can be disregarded, the parameters ν_0, D_0, and λ_0 will have changed to $\nu(l_0)$, $D(l_0)$ and $\lambda(l_0)$. It is at this point, practically speaking, that our present explicit analysis starts.

Model C is more realistic than models A and B because the fluid is stirred only in a narrow band of wave numbers. Although somewhat idealized, it nevertheless represents a good example of the kind of "universality" discussed in the above paragraph. The large-distance and long-time properties of model C are just those of model A.

To demonstrate this, we use the renormalization group developed for model B to "integrate out" the pulse of force in $\vec{k}$ space. Repeated elimination of degrees of freedom occupying shells in $\vec{k}$ space gradually removes the constant part of the force autocorrelation. No new contribution to the constant part of the force-force correlations are generated at small k because, as discussed in Sec. III E, graphs like that in Fig. 1(b) contribute only k^2 corrections to the renormalized force autocorrelations. Recursion relations such as (3.71)–(3.73) need only be integrated until $l = l_0 = \ln(\Lambda/\tilde{\Lambda})$, when the lower edge of the pulse is reached. At this point, the system is described by renormalized equations of motion in which "partially dressed" couplings $\nu(l_0)$ and $\lambda(l_0)$ appear.

The crucial feature is that the force-force correlations no longer contain a constant part, and behave instead as k^2 for small k. Although this k^2 term was irrelevant in the analysis of model B, it now dominates the infrared behavior. No nonanalytic terms such as $|\vec{k}|$ or $k^{3/2}$ can appear at this point because the (analytic) recursion relations have only been integrated a finite amount of "time" $l_0 = \ln(\Lambda/\tilde{\Lambda})$. Model C now resembles model A, and can by analyzed by the methods developed in Sec. III B. In particular, the infrared properties of the two models should be identical.

These conclusions should not depend on the rather special rectangular shape of $D(k)$. We expect that any force whose autocorrelation is cut off both above and below will generate dynamics falling into the universality class exemplified by model A.

G. Results to all orders in ϵ

The reader may have noted that the recursion relations such as (3.23) and (3.73) derived for "convective" coupling constants in this Section are all extremely simple. No nonlinear terms $[\mathcal{O}(\lambda^3)]$ appear on the right-hand side of the recursion formulas for $\lambda(l)$. This feature is the reason, for example, for the simple canonical exponents $(t^{-d/2})$ which characterize long-time tail phenomena for model A above 2 dimensions, as well as the simple exponents $\frac{1}{2}$ of logarithmic corrections at $d = 2$. For model A this feature is quite general, a consequence of the Galilean invariance (2.18) of the underlying equations of motion which is also a symmetry of the shell integration. A graphical proof will be given in Sec. IV.

Accepting this multiplicative renormalization of $\lambda(l)$, results such as

$$z = 2 - \tfrac{1}{2}\epsilon \quad (\epsilon \equiv 2 - d > 0) \qquad (3.81)$$

for model A and

$$z = 2 - \tfrac{1}{3}\epsilon \quad (\epsilon \equiv 4 - d > 0) \qquad (3.82)$$

for model B appear to be correct to all orders in ϵ. The derivation of such results by means of recursion relations was sketched for model A in Ref. 14, and is given in detail for a problem in dynamic critical phenomena by Halperin, Hohenberg, and Siggia.[31] We will not pause to repeat such a demonstration here, but instead refer the reader to the graphical treatment in Sec. IV. As stated in that Section, we have been unsuccessful in producing a graphical proof that the result (3.82) for model B is correct to all orders.

IV. DIRECT GRAPHICAL ANALYSIS

At this point we demonstrate that results obtained in the previous Section can be quickly and efficiently derived using a direct graphical approach. Although the techniques employed here are perhaps more familiar than the recursion relation formalism of Sec. III, they do in fact rely on renormalization group ideas, such as Wilson's Feynman graph approach,[2] and the parquet graph resummation method.[32] They allow a convincing demonstration that results obtained for model A are in fact valid to all orders in $\epsilon \equiv 2-d$, and permit predictions to be made about the infrared properties of Burger's equation in one dimension. Part of the utility of this graphical approach rests on the Ward identity proved in Appendix B. As mentioned in the Introduction, we expect the recursion analysis to be of more utility in situations where Ward identities do not produce such enormous simplifications.

A. Model A

We first discuss the small k, small ω properties of the response function $G_R(k,\omega) \equiv \langle \delta v_i / \delta f_i \rangle (\vec{k},\omega)$ and the correlation function

$$G(k,\omega) = \mathrm{Tr} G_{ij}(\vec{k},\omega) \tag{4.1}$$

appropriate to model A. An analogy with critical phenomena suggests that these functions can be written in a scaled form,

$$\begin{aligned} G^{-1}(k,\omega) &= k^{2-\eta} g(\omega/k^z), \\ G_R^{-1}(k,\omega) &= k^{2-\eta} g_R(\omega/k^z), \end{aligned} \tag{4.2}$$

where the exponents η and z are to be determined. The fluctuation-dissipation theorem discussed in Appendix B requires that G^{-1} and G_R^{-1} scale with identical exponents η and z. In a linear treatment of the fluctuations, these two functions are simply given by

$$\begin{aligned} G_0(k,\omega) &= 2D_0 k^2/[\omega^2 + \nu_0^2 k^4], \\ G_{0R}(k,\omega) &= [-i\omega + \nu_0 k^2]^{-1}, \end{aligned} \tag{4.3}$$

which satisfy (4.2) with $\eta = 0$ and $z = 2$.

For model A at least, these exponents can be determined more generally by the following simple arguments:

(i) The *equal time* velocity fluctuations are given by

$$\frac{\langle v_i(\vec{k},t) v_j(\vec{k}',t)\rangle}{(2\pi)^d \delta(\vec{k}+\vec{k}')} = \int_{-\infty}^{+\infty} G(k,\omega)\, d\omega/2\pi = D_0/\nu_0, \tag{4.4}$$

where this last result follows from the Gaussian distribution of these fluctuations [Eq. (B1)]. This leads immediately to an exponent relation,

$$z = 2 - \eta. \tag{4.5}$$

(ii) A second relation between η and z follows from simple power counting arguments. In Appendix B it is shown that vertex corrections vanish for small k and ω. The requirement that perturbation theory be consistent with (4.2) and this result leads to an additional relation,

$$3\eta + z = \max(2, 4-d). \tag{4.6}$$

We thus have two relations for the exponents η and z. On substituting the values $\eta = 0, z = 2$ appropriate to linearized hydrodynamics into (4.6) we see that $d = 2$ is a special case. The following results, which agree with those of Sec. III, are then obtained:

(1) $d < 2$.

$$\eta = \tfrac{1}{2}(2-d), \quad z = 2 - \tfrac{1}{2}(2-d), \tag{4.7}$$

and the renormalized viscosity is singular,

$$\begin{aligned} \nu_R(k, \omega=0) &\sim k^{-(1/2)(2-d)}, \\ \nu_R(k=0,\omega) &\sim (1/\omega)^{(2-d)/(2+d)}. \end{aligned} \tag{4.8}$$

The correlation and response functions can be determined as expansions in powers of $\epsilon = 2-d$.[14]

(2) $d > 2$. To leading order the exponents and correlation and response functions in Eq. (4.2) are given by linearized hydrodynamics. There are, however, nontrivial corrections of the form discussed in Sec. III. These corrections may be determined by calculating the self-energy contributions to $G^{-1}(k,\omega)$ and $G_R^{-1}(k,\omega)$. In three dimensions, the renormalized viscosity appearing in (3.37) is then given by

$$\nu_R(0,\omega) = \nu + [7i\chi/120\pi\nu](i\omega/2\nu)^{1/2} + \Theta(\omega), \tag{4.9}$$

$$\nu_R(k,0) = \nu - (1+\tfrac{3}{4}\pi)(\chi/96\pi\nu)k + \Theta(k^2), \tag{4.10}$$

where as in Sec. III various constant terms have been absorbed into the measured viscosity ν, and $\chi = D_0/\nu_0 = k_B T/\rho$.

(3) $d = 2$. This case requires special consideration since logarithmic corrections arise. However, we can apply the parquet method used in critical phenomena[32] by writing a self-consistent equation for the renormalized viscosity. This equation can be determined as an expansion in the parameter $1/\nu_R$ which is logarithmically small:

$$\frac{\partial}{\partial k}\nu_R(k,0) = -\frac{\chi}{16\pi}\frac{1}{k\nu_R(k,0)} + \Theta(1/\nu_R^3). \tag{4.11}$$

Imposing the boundary condition $\nu_R(k=\Lambda,0) = \nu_0$, this equation integrates to

$$\nu_R(k,0) = \nu_0[1 + (\chi/8\pi\nu_0^2)\ln(\Lambda/k)]^{1/2}. \tag{4.12}$$

An analogous calculation gives

$$\nu_R(0,\omega)=\nu_0[1+(\chi/16\pi\nu_0^2)\ln(\nu_0\Lambda^2/\omega)]^{1/2}. \quad (4.13)$$

These results agree, of course, with those obtained in Sec. III by different methods.

B. Burger's equation in one dimension

The results described above can be extended to the Burger's equation model described in Sec. II B, but only in one dimension. It is only in one dimension that a fluctuation-dissipation theorem relates the response and correlation functions generated by Burger's equation in a simple way. Just as for model A, one can show that the vertex corrections to Burger's equation are negligible in the infrared limit. It follows, repeating the arguments of the previous subsection, that

$$\eta=\tfrac{1}{2}, \quad z=\tfrac{3}{2} \quad (4.14)$$

for Burger's equation in one dimension.

C. Model B

There is no obvious fluctuation-dissipation theorem for this model, which complicates the analysis. We might again expect that the response and correlation functions scale, and postulate the functional forms

$$G^{-1}(k,\omega)=k^z g(\omega/k^z), \quad G_R^{-1}(k,\omega)=k^{2-\eta}g_R(\omega/k^z). \quad (4.15)$$

Linearized hydrodynamics gives the results $\sigma=\eta=0$, $z=2$, and

$$G_0(k,\omega)=2D_0/[\omega^2+\nu_0^2k^4], \quad G_{0R}(k,\omega)=1/[-i\omega+\nu_0k^2]. \quad (4.16)$$

We determine the exponents to leading order in $\epsilon=4-d$ by the following arguments:

(i) In the limit $k\to 0$ we have $G_R^{-1}(0,\omega)=-i\omega$ which requires again that

$$z=2-\eta. \quad (4.17)$$

(ii) Arguments similar to those used in Appendix B show that the leading vertex corrections for this model vanish. We have not been able to extend this result to higher orders.

(iii) Neglecting vertex renormalization, power counting arguments relate the exponents η, z, and σ,

$$4\eta+z+\sigma=\max(2,6-d). \quad (4.18)$$

The critical dimension in this case is $d=4$. A detailed calculation of the self-energy to first order in $\epsilon=4-d$ shows that σ is at least of order ϵ^2. Thus the above relations give, to leading order in $\epsilon=4-d$,

$$\eta=\tfrac{1}{3}\epsilon+\mathcal{O}(\epsilon^2), \quad z=2-\tfrac{1}{3}\epsilon+\mathcal{O}(\epsilon^2). \quad (4.19)$$

V. SUMMARY

We have applied renormalization group methods useful in studies of dynamic critical phenomena to the large-distance, long-time properties of a randomly stirred fluid. Long-time tail phenomena and the large eddy properties of the forced Navier-Stokes equation are understood in terms of an attractive hydrodynamic fixed point above two dimensions. A stable, nontrivial fixed point appears below $d=2$. The slow approach to the hydrodynamic fixed point in exactly two dimensions leads to logarithmic corrections to conventional hydrodynamics. Although the physical significance of an incompressible fluid in less than two dimensions is unclear, we have produced a model, model B, for which hydrodynamics breaks down below *four* dimensions. This breakdown is described in terms of scaling laws and exponents, and is associated with a nontrivial fixed point.

Most results obtained for a fluid near thermal equilibrium were derived previously using the mode-coupling approximation.[19] The large eddy properties of a randomly stirred fluid presumably follow from the analogous approximation scheme in turbulence theory, Kraichnan's direct interaction approximation.[33] Many of the results obtained in renormalization studies of critical dynamics[4] were also anticipated by mode coupling theories.[34]

The advantage of the approach taken here is that it renders these essentially uncontrolled approximations systematic. Specifically, we find that simple low-order perturbation schemes (such as the mode-coupling or DIA integral equations) are adequate provided we are interested only in the small k, small ω properties of the correlations. Conversely, our results suggest that these approximations would be rather poorly suited to treat the *ultraviolet* behavior of a fluid. Heuristically at least, one can imagine running the recursion formulas presented here backwards in an attempt to study large wave-number properties. However, the recursion relations derived here [see, e.g., Eq. (3.26)] indicate that the effective coupling constant $\bar{\lambda}(l)$ will become *larger*, not smaller, at these wave numbers. Weak-coupling perturbation schemes simply will not work.

Accompanying the formal manipulations entering a renormalization group transformation are Wilson's ideas[16] about irrelevant variables. To test the importance of a particular term in a fluid equation, we simply study its recursion formula in the vicinity of the fixed point of interest. If this term decays rapidly to zero, it can be neglected in an analysis of the infrared long-time behavior. Similar reasoning leads us to neglect corrections to the Gaussian character of the random force in the lim-

it. To convincingly exclude non-Gaussian or velocity dependant contributions to the force using microscopic arguments alone appears to be rather difficult.

We would also like to point out the essential simplicity of the considerations presented here. In particular, the recursion relations (3.21)–(3.23) for Model A can be written down on general grounds, viz.: (i) $D_I > D_0$ is a stability requirement, (ii) $D_I/D_0 = \nu_I/\nu_0$ is a consequence of the fluctuation-dissipation theorem, and (iii) $\lambda_I = \lambda_0$ follows from Galilean invariance in conjunction with momentum conservation. Thus the only feature of Eqs. (3.21)–(3.23) which requires explicit calculation to order λ^2, is the magnitude of the coefficient A_d; in other words the amplitude of long-time tails, while their leading exponents are universally determined.

In conclusion, we believe there are advantages in describing fluctuating hydrodynamics in terms of fixed points and recursion flows. It is our hope that the techniques described here will be of utility in attacking other problems in fluid mechanics or in irreversible statistical mechanics.

APPENDIX A: EXPLICIT CALCULATION OF A RECURSION FORMULA

As an illustration of the analysis developed in Sec. III, we extract a recursion relation for model A from the graphs shown in Fig. 1(a). The procedure is very similar to that sketched by Ma and Mazenko,[8] but is complicated by a proliferation of indices appearing on transverse projection operators. The indices and momenta accompanying the Feynman graph of Fig. 1(a) are shown in Fig. 3, together with the meaning of its constituent elements.

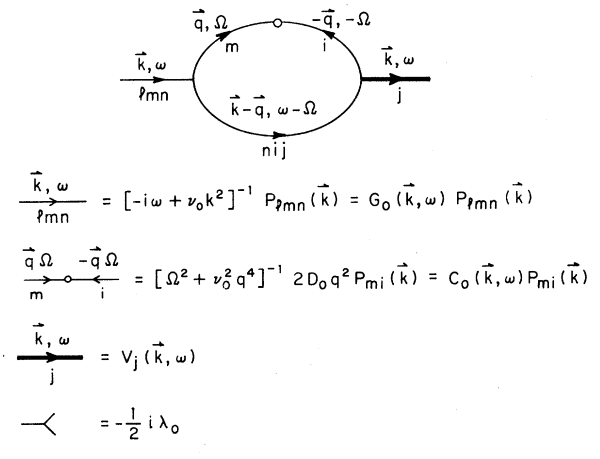

FIG. 3. The momenta, frequencies, and vector indices accompanying the Feynman graph shown in Fig. 1(a). The meanings of the various constituent elements are also indicated.

Writing out the algebraic equation associated with Fig. 1(a), we have

$$v_i^<(\vec{k},\omega) = G_0(k,\omega) f_i^<(\vec{k},\omega) + 4G_0(k,\omega)(\tfrac{1}{2}i\lambda_0)^2 \times P_{lmn}(\vec{k}) I_{mnj}(\vec{k},\omega) v_j^<(\vec{k},\omega) + \cdots , \quad \text{(A1)}$$

where

$$I_{mnj}(\vec{k},\omega) = \int_{q\Omega}^{>} P_{nij}(\vec{k}-\vec{q}) G_0(\vec{k}-\vec{q},\omega-\Omega) \times C_0(q,\Omega) P_{im}(\vec{q}) . \quad \text{(A2)}$$

A combinatorial factor 4 is associated with the graph of Fig. 3, and there is an implied summation over repeated indices in (A1) and (A2). The symbol $\int_{q\Omega}^{>}$ means $(2\pi)^{-(d+1)} \int_{-\infty}^{\infty} d\Omega \int d^d q$, where the momentum integrals are restricted to the domain $e^{-l} < |\vec{q}| < 1$, $e^{-l} < |\vec{k}-\vec{q}| < 1$. The upper momentum cutoff has been fixed, for convenience, at unity.

Equation (A1) can be rearranged to give an equation of motion of the form

$$[-i\omega + k^2 \nu_I(k,\omega)] v_i^<(\vec{k},\omega) = f_i^<(\vec{k},\omega) + \cdots , \quad \text{(A3)}$$

where $\nu_I(k,\omega) = \nu_0 + \Delta\nu_I(k,\omega)$, with

$$k^2 \Delta\nu_I(k,\omega) P_{ij}(\vec{k}) = \lambda_0^2 P_{lmn}(\vec{k}) I_{mnj}(\vec{k},\omega) . \quad \text{(A4)}$$

Thus the diagram in Fig. 3 renormalizes the viscosity. The appearance of the projection operator on the left-hand side of this equation is an obvious consequence of symmetry.

The frequency integral in (A2) is readily done, with the result

$$k^2 \Delta\nu_I(k,\omega) P_{ij}(\vec{k}) = \lambda_0^2 (D_0/\nu_0) P_{lmn}(\vec{k}) k_i \times \int^{>} \frac{d^d q}{(2\pi)^d} \frac{P_{nj}(\tfrac{1}{2}\vec{k}-\vec{q}) P_{mi}(\tfrac{1}{2}\vec{k}+\vec{q})}{-i\omega + 2\nu_0 q^2 + \tfrac{1}{2}\nu_0 k^2} , \quad \text{(A5)}$$

where we have made a convenient change of variables, namely $\vec{q} \to \vec{q} + \frac{1}{2}\vec{k}$, and used the properties of the projection operators. It is now apparent that, for small k and ω, we can evaluate the integral in (A5) at $k = 0$ and $\omega = 0$. The momentum integral is then restricted to the spherical shell $e^{-l} < |\vec{q}| < 1$, and the angular average of the projection operators is evaluated straightforwardly. We obtain finally

$$\Delta\nu_I(0,0) = \frac{\lambda_0^2}{2} \frac{D_0}{\nu_0^2} \frac{d^2-2}{d^2+2d} \frac{s_d}{(2\pi)^d} \int_{e^{-l}}^{1} \frac{q^{d-1} dq}{q^2} , \quad \text{(A6)}$$

where $s_d = 2\pi^{(1/2)d}/\Gamma(\frac{1}{2}d)$. Thus the "intermediate" viscosity $\nu_I = \nu_0 + \Delta\nu_I(0,0)$ is given by (3.10), with

$$A_d = \frac{1}{2}\left(\frac{d^2-2}{d^2+2d}\right)\frac{s_d}{(2\pi)^d} . \quad \text{(A7)}$$

It is a simple matter, in principle, to work out

graph rules that describe the extension of this second-order calculation to higher orders. We will not do so here but mention that (i) the sum of all $1PI$ diagrams with one incoming and one outgoing arrow renormalizes the viscosity ν_0, (ii) the sum of all $1PI$ diagrams with two incoming arrows renormalized the force strength D_0, and (iii) the sum of all $1PI$ diagrams with one incoming and two outgoing arrows renormalizes the coupling constant λ_0. $1PI$ diagrams are those which do not fall apart if a single line is cut, and they have no factors G_0 associated with the external lines. The momentum integrals over the internal lines are all restricted to the shell $e^{-l} < |\vec{q}| < 1$.

Finally, we note that if we integrate $\vec{q}$ in (A5) over the full momentum space, $0 < |\vec{q}| < 1 (= \Lambda)$, we obtain the quantity $\nu_R(k,\omega) - \nu_0$ defined by Eq. (3.37), to order $\bar{\lambda}_0^2$. The calculation of long-time tail corrections like (3.40) is based on (A5), in conjunction with the homogeneity relation (3.38) for ν_R. Equation (A5) is, of course, then just the standard-mode coupling formula.

APPENDIX B: FLUCTUATION-DISSIPATION THEOREM AND VERTEX CORRECTIONS IN MODEL A

A perturbation theory applicable to the Navier-Stokes equation has been developed by Martin, Siggia, and Rose (MSR)[9] and others. The theory involves correlation functions $[G(k,\omega)]$, response functions $[G_R(k,\omega)]$ and vertex functions (Γ), and in general all these must be worked out. In the case of model A a fluctuation-dissipation theorem (FDT) exists and the perturbation theory can be simplified considerably.

Several classes of classical processes for which FDT's exist have been discussed by Deker and Haake.[35] Model A exhibits detailed balance, and the irreversible terms in the equation of motion are linear in the velocity; it thus corresponds to the second class considered by Deker and Haake. It is then easy to show that the equal-time velocity fluctuations are determined by the Gaussian distribution

$$P_0 \sim \exp\left[-(\nu_0/2D_0)\sum_{\vec{k}} v_i(\vec{k})v_i^*(\vec{k})\right] \tag{B1}$$

and thus show no corrections in any dimension. In particular this result also applies to Burger's equation (2.8) with the forcing function (2.10).

The FDT takes the form

$$G(k,\omega) = (D_0/\nu_0)[G_R(k,\omega) + G_R^*(k,\omega)] \tag{B2}$$

connecting the correlation and response functions. This relation can also be written as a relation connecting the self-energies. The self-energies are defined by

$$G(k,\omega) = G_R(k,\omega)[2D_0k^2 + \Sigma(k,\omega)]G_R^*(k,\omega)\,, \tag{B3}$$

$$G_R(k,\omega) = [-i\omega + \nu_0 k^2 - \Sigma_R(k,\omega)]^{-1}\,. \tag{B4}$$

Equation (B2) becomes

$$\Sigma(k,\omega) = -(D_0/\nu_0)[\Sigma_R(k,\omega) + \Sigma_R^*(k,\omega)]\,. \tag{B5}$$

The perturbation theory of MSR can be further simplified for model A by noting that vertex corrections vanish in the infrared limit. This result is essentially a consequence of Galilean invariance and is easily proved to all orders in perturbation theory. The theory of MSR involves three vertices. However, in the case when the steady-state distribution is Gaussian it has been shown by Kawasaki[36] that only one type of vertex need be considered. In the notation of MSR one then shows that the vertex $\Gamma^{(1)}_{ijm}(\vec{k}_1,\vec{k}_2;\, t_1,t_2,t_3)$ reduces to its bare value $\gamma_{ijm}(\vec{k}_1+\vec{k}_2)\delta(t_1-t_2)\delta(t_1-t_3)$ when the external momenta $\vec{k}_1$ and $\vec{k}_2$ are small. The bare interaction has the form

$$\gamma_{ijm}(\vec{k}) = -ik_m(\delta_{ij} - \hat{k}_i\hat{k}_j) - ik_j(\delta_{im} - \hat{k}_i\hat{k}_m)\,. \tag{B6}$$

Typical diagrams contributing to Γ are shown in Figs. 4(a) and 4(b). The method of proof employed here is similar to some methods used by Deker and Haake. We need the following:

(a) We consider each diagram of perturbation theory at fixed times of the vertices. We note that t_1 is always the latest time in a diagram because the t_1 vertex is connected to every other vertex through a series of retarded response functions.

(b) On the internal lines of a diagram we neglect the external momenta $\vec{k}_1$ and $\vec{k}_2$.

(c) The bare interaction has the form (B6) and one notices that the pressure terms will not contribute if the vertex is an internal one (i.e., has no external lines attached), because of the incompressibility condition. Thus, what enters is

$$\gamma'_{ijm}(\vec{k}) = -ik_m\delta_{ij} - ik_j\delta_{im}\,. \tag{B7}$$

(d) We use the FDT (B2) to replace all correlation functions by retarded or advanced response functions depending on the time labels attached to the

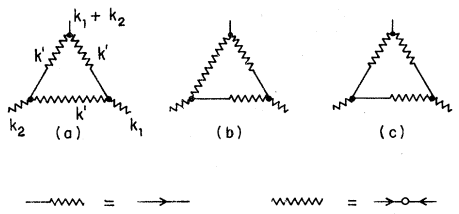

FIG. 4. The two third-order diagrams (a) and (b) in case $t_1 > t_2 > t_3$. Response functions G_R are denoted by ∿∿— and correlation functions G by ∿∿∿. In (a) we have neglected the external mometa on the internal lines of the diagram.

two vertices involved.

Making use of (a)–(d) alone it is possible to show that all the perturbation terms (except the bare interaction) for $\Gamma^{(1)}$ cancel. An example is given in Figs. 4(a) and 4(b). With the time order $t_1 > t_2 > t_3$ we use (d) to convert both these diagrams into the form shown in Fig. 4(c). The lower left vertex has the form $\gamma'_{ijm}(-\vec{k}')$ in 4(a) and $\gamma'_{ijm}(\vec{k}')$ in 4(b), the rest of the diagram being the same in both cases. As $\gamma'_{ijm}(-\vec{k}') + \gamma'_{ijm}(\vec{k}') = 0$, the sum of 4(a) and 4(b) vanishes.

In general we look at the vertex with the earliest time label. If this is an internal vertex there will be three diagrams which are distinct in that this vertex appears in the three different orientations, the rest of the diagrams being the same. When we use the FDT the three vertex ends become equivalent, and the sum of the three contributions will thus be proportional to (omitting the pressure terms) $\gamma'_{ijm}(\vec{k}) + \gamma'_{ijm}(\vec{k}_1) + \gamma'_{ijm}(\vec{k}_2) = 0$. If the vertex with the earliest time is an external one (either t_2 or t_3) it can appear as in Figs. 4(a) and 4(b), the rest of the diagram being the same. The sum of the two contributions will cancel as in the example of Fig. 4.

Thus in the limit of small external momenta the vertex $\Gamma^{(1)}$ reduces to its bare value (B6), with no corrections due to renormalization. This result also applies to Burger's equation in one dimension.

ACKNOWLEDGMENTS

This work was carried out independently at Temple University and at Harvard University. We would like to thank P. C. Martin and E. Siggia for helpful discussions. Dieter Forster would also like to acknowledge stimulating and useful conversations with M. Droz, M. Green, and J. D. Gunton. David R. Nelson is particularly indebted to U. Frisch for helpful comments.

---

*Work supported in part by the National Science Foundation under Grant No. DMR 76-11974.

†Work supported by a Junior Fellowship, Harvard Society of Fellows, and in part by the National Science Foundation under Grant No. DMR 72-02977 A03.

‡Work supported in part by the Rutgers University Research Council and the National Science Foundation under Contract No. GH 38458. Part of this work was done at Harvard University whose hospitality is gratefully acknowledged.

[1]See, e.g., S. Coleman, in *Properties of the Fundamental Interactions* 1971, edited by A. Zichichi (Editrice Compositori, Bologna, Italy, 1972); H. D. Politzer, Phys. Rep. 14C, 131 (1974).

[2]K. G. Wilson and J. Kogut, Phys. Rep. 12C, 77 (1974); M. E. Fisher, Rev. Mod. Phys. 42, 597 (1974); S.-K. Ma, Rev. Mod. Phys. 45, 589 (1973). See also S.-K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, Reading, Mass., 1976).

[3]For an application to the Kondo problem, see K. G. Wilson, Rev. Mod. Phys. 47, 773 (1975).

[4]See, e.g., B. I. Halperin, P. C. Hohenberg, and S. K. Ma, Phys. Rev. Lett. 29, 1548 (1972); Phys. Rev. B 10, 137 (1974); B. I. Halperin, P. C. Hohenberg, and E. Siggia, Phys. Rev. Lett. 28, 548 (1974); S. Ma and G. Mazenko, Phys. Rev. Lett. 33, 1384 (1974).

[5]K. Kawasaki, Ann. Phys. (N.Y.) 61, 1 (1970).

[6]In particular, we restrict ourselves here to *incomprehensible* fluids. The compressibility of a fluid diverges at the critical point.

[7]P. G. Saffman, J. Fluid Mech. 27, 581 (1967).

[8]S.-K. Ma and G. Mazenko, Phys. Rev. B 11, 4077 (1975).

[9]P. C. Martin, E. G. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973).

[10]This has been done for spin systems by R. Bausch, H. K. Jansen, and H. Wagner (Institut für Festkörperforschung, KFA Jülich preprint).

[11]A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, Vol. II (M.I.T., Cambridge, 1975).

[12]An introduction to modern turbulence theory is given by S. A. Orszag, in *Hydrodynamics*, edited by R. Balian (Gordon and Breach, New York, 1974).

[13]G. K. Batchelor and I. Proudman, Philos. Trans. R. Soc. A 248, 369 (1956).

[14]Results for model A have appeared elsewhere, D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. Lett. 36, 867 (1976).

[15]This interpretation is due to P. C. Martin.

[16]K. G. Wilson, Phys. Rev. B 4, 3174 (1971).

[17]U. Frisch, M. Lesieur, and P. L. Sulem, Phys. Rev. Lett. 37, 895 (1976).

[18]A discussion of Burger's equation is given by P. G. Saffman, in *Topics in Nonlinear Physics*, edited by N. J. Zabusky (Springer-Verlag, New York, 1968).

[19]B. J. Alder and T. E. Wainwright, Phys. Rev. A 1, 18 (1970); M. H. Ernst, E. H. Hauge, and J. M. J. van Leeuwen, Phys. Rev. Lett. 25, 1254 (1970) and Phys. Rev. A 4, 2055 (1971); J. R. Dorfman and E. G. D. Cohen, Phys. Rev. Lett. 25, 1257 (1970) and Phys. Rev. A 6, 776 (1972) and Phys. Rev. A 12, 292 (1975). For two recent reviews and many additional references, see Y. Pomeau and P. Résibois, Phys. Rep. 19C, 64 (1975), and R. Dorfman, in *Fundamental Problems in Statistical Mechanics*, edited by E. G. D. Cohen (North-Holland/American Elsevier, Amsterdam, 1975).

[20]R. Zwanzig, in *Lectures in Theoretical Physics*, edited by W. R. Brittin (Interscience, New York, 1961) and Phys. Rev. 124, 983 (1961); H. Mori, Prog. Theor. Phys. (Kyoto) 33, 423 (1965).

[21]L. P. Kadanoff and P. C. Martin, Ann. Phys. (N.Y.) 24, 419 (1963).

[22]See, e.g., R. Zwanzig, in *Proceedings of the I.U.P.A.P. Conference on Statistical Mechanics*, edited by S. A. Rice, K. Freed, and J. Light (University of Chicago Press, Chicago, 1972).

[23]See, e.g., K. S. J. Nordholm and R. Zwanzig, J. Stat.

Phys. 11, 143 (1974); L. S. García-Colín and R. M. Velasco, Phys. Rev. A 12, 646 (1975).

[24]Except in an equilibrium fluid where, as outlined in Sec. IIC, the cutoff is considered part of the equations of motion themselves, with high-k modes treated as noise.

[25]Strictly speaking, the average over $f^>$ should be done at constant $v^<$, rather than $f^<$.

[26]See, e.g., R. H. Kraichnan, J. Fluid Mech. 5, 497 (1959); W. W. Wyld, Jr., Ann. Phys. (N.Y.) 14, 143 (1961), and Ref. 10.

[27]F. J. Wegner, Phys. Rev. B 5, 4529 (1972).

[28]S. F. Edwards and W. D. McComb, J. Phys. A 2, 157 (1969).

[29]As shown in Ref. 9, renormalized dynamical theories in general involve *three* distinct quadratic nonlinearities in the equations of motion. However, the nonlinearities considered in Sec. III dominate the infrared behavior. The remaining nonlinearities (which do not appear in the unrenormalized equations of motion) involve higher powers of k and are irrelevant variables.

[30]In precisely two dimensions the equilibrium distribution of a fluid is expected to correspond to constant energy $\sum_k \vec{v}_k \cdot \vec{v}_{-k}$ *and* constant vorticity $\sum_k k^2 \vec{v}_k \cdot \vec{v}_{-k}$. This would be engendered by a forcing function $D(k) = \nu k^2/[\beta + \gamma k^2]$ as was pointed out to us by Herbert Wagner. The difference to the forcing function $D_0 k^2$ considered for model A is irrelevant for the infrared properties.

[31]B. I. Halperin, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. B 13, 1299 (1976).

[32]Larkin and Khmelnitski, Zh. Eksp. Teor. Fiz. 56, 2087 (1969) [Sov. Phys.-JETP 29, 1123 (1969)]; and T. Tsuneto and E. Abrahams, Phys. Rev. Lett. 30, 217 (1973).

[33]R. H. Kraichman (private communication); see Ref. 26 and R. H. Kraichnan, Adv. Math. 16, 305 (1975).

[34]The relation between renormalization group and mode-coupling theories of critical dynamics is discussed by K. Kawasaki and J. Gunton, Phys. Rev. B 13, 4658 (1976).

[35]U. Deker and F. Haake, Phys. Rev. A 11, 2043 (1975).

[36]K. Kawasaki, Prog. Theor. Phys. 52, 1527 (1971).